

## II — Principal Axes of a Euclidean Cloud

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This text is adapted from the chapter 2 of the monograph *Multiple Correspondence Analysis* (QASS series n°163, SAGE, 2010)

## II.1 Basic Geometric Notions

Elements of a geometric space: *points, line, plane.*

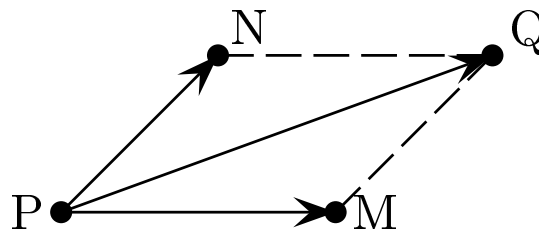
— *Affine notions*: alignment, direction and barycenter.

Couple of points (P, M), or *dipole*  $\longrightarrow$  *vecteur*  $\overrightarrow{PM}$

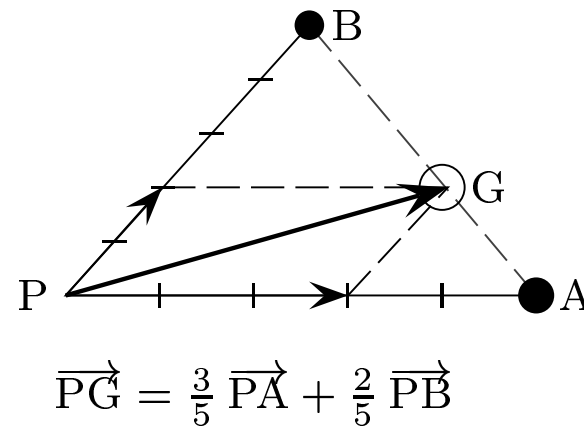
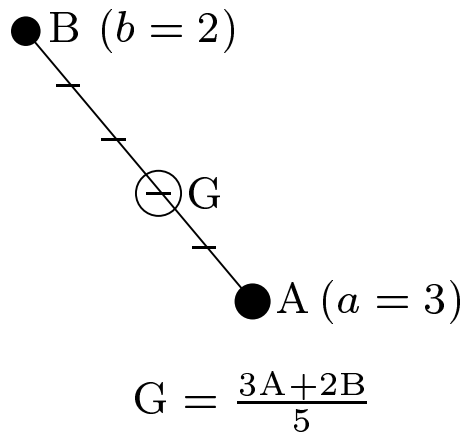
• The deviation from point P to point M is  $M - P$  (“terminal minus initial”), that is,  $\overrightarrow{PM}$ .

Deviations add up vectorially: sum of vectors by *parallelogram rule*

$$\overrightarrow{PM} + \overrightarrow{PN} = \overrightarrow{PQ}$$



- Barycenter of a dipole



Barycenter = *weighted average of points*:  $G = \frac{aA + bB}{a + b}$

— *Metric notions*: distances and angles.

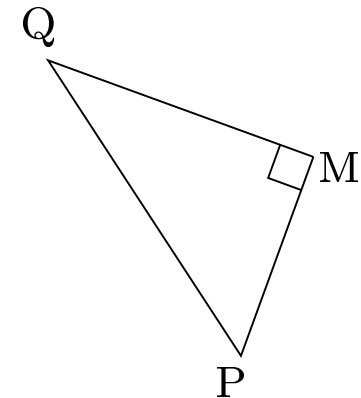
Triangle inequality:  $PQ \leq PM + MQ$

*Pythagorean theorem*:

If PM and MQ are perpendicular then:

$$(PM)^2 + (MQ)^2 = (PQ)^2$$

(triangle MPQ with right angle at M),



## II.2 Cloud of Points

### Target example

Figure 1. Target example (10 points)

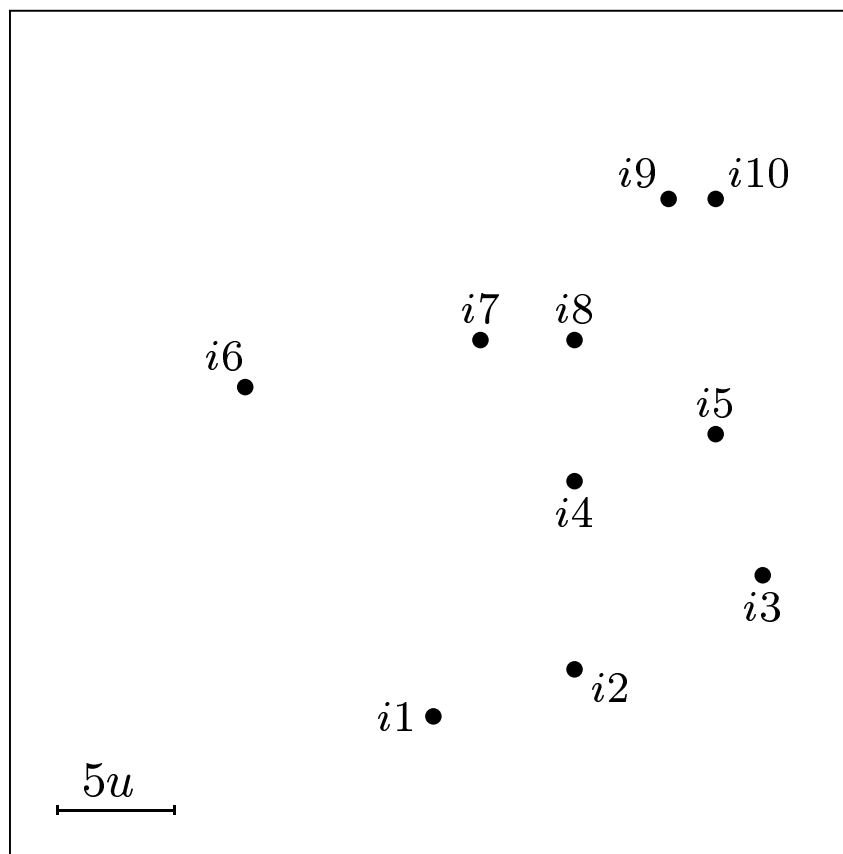
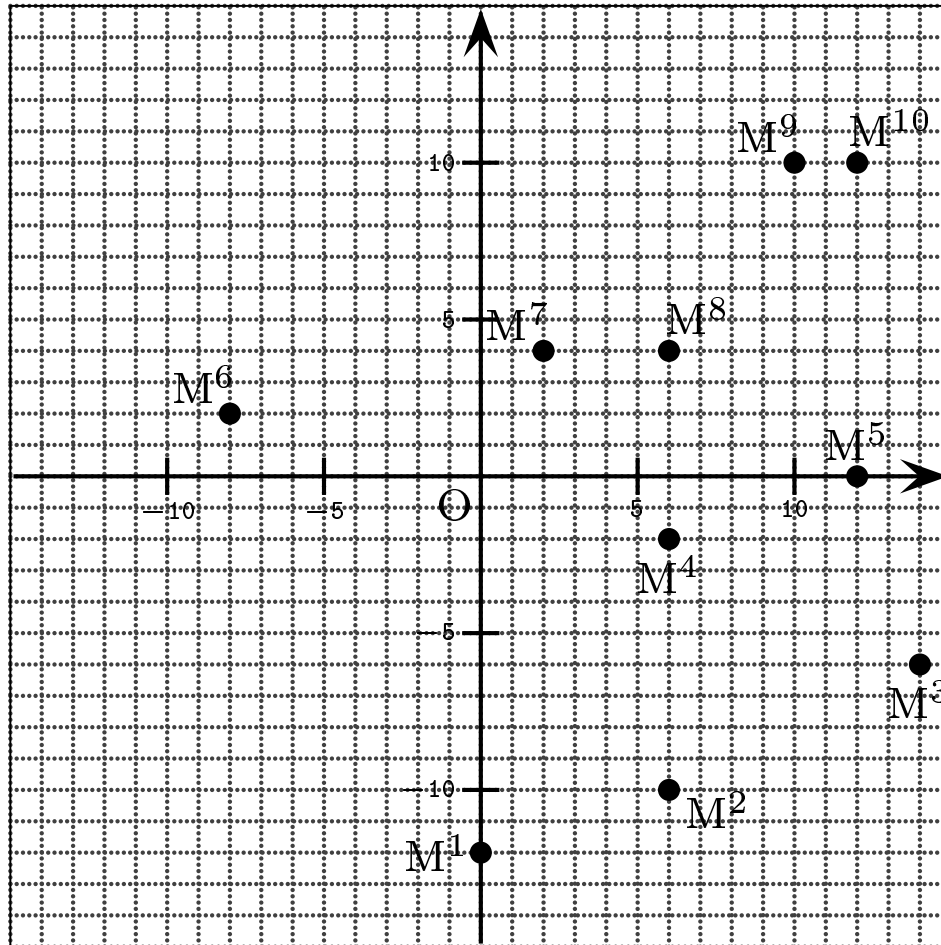


Figure 1bis. Cloud of 10 points with origine-point O and initial axes

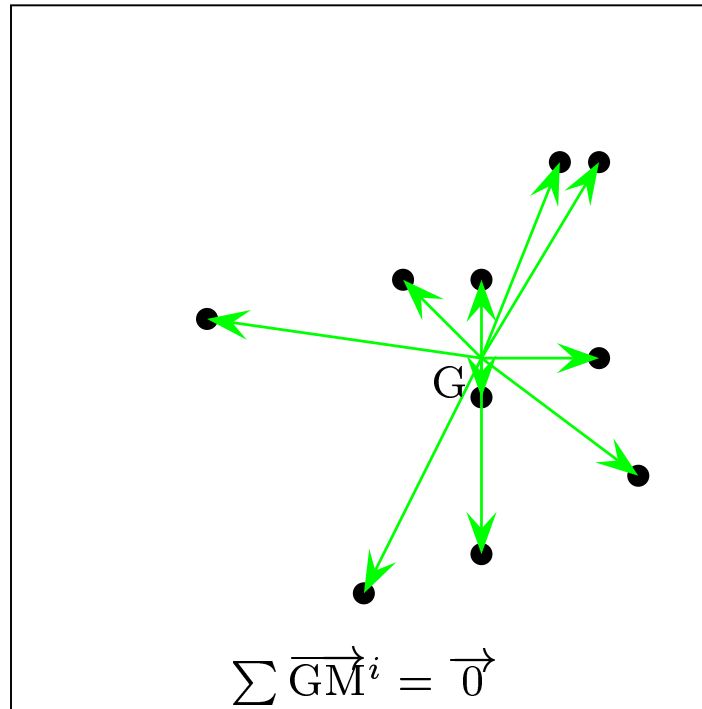
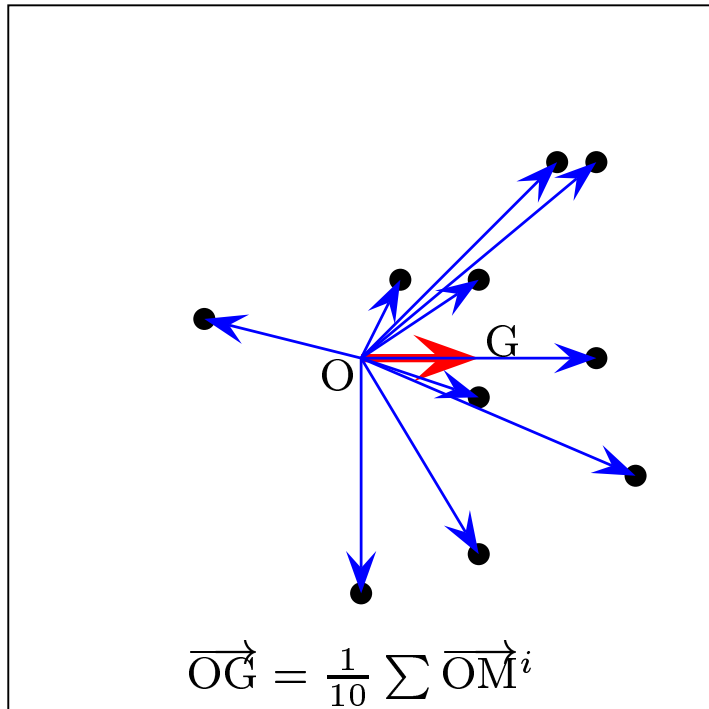


Initial coordinates

	$x_1$	$x_2$	weight
M <sup>1</sup>	0	-12	1
M <sup>2</sup>	6	-10	1
M <sup>3</sup>	14	-6	1
M <sup>4</sup>	6	-2	1
M <sup>5</sup>	12	0	1
M <sup>6</sup>	-8	2	1
M <sup>7</sup>	2	4	1
M <sup>8</sup>	6	4	1
M <sup>9</sup>	10	10	1
M <sup>10</sup>	12	10	1
Means	6	0	[10]
Variances	40	52	
Covariance		+ 8	

*Mean point:* point G

$$\overrightarrow{OG} = \sum p_i \overrightarrow{OM}^i \qquad \sum p_i \overrightarrow{GM}^i = \overrightarrow{0} \text{ (barycentric property).}$$



*Target Example:*  $p_i = \frac{1}{n}$  [= 0.1]

*Variance of a cloud :*

$$V_{\text{nuage}} = \sum p_i (\text{GM}^i)^2$$

(see Benzécri 1992, p.93)

*Property.* In rectangular axes, the variance of the cloud is the sum of the variances of the coordinate variables.

*Contribution of point  $M^i$ :*

$$\text{Ctr}_i = \frac{p_i(\text{GM}^i)^2}{V_{\text{nuage}}}$$

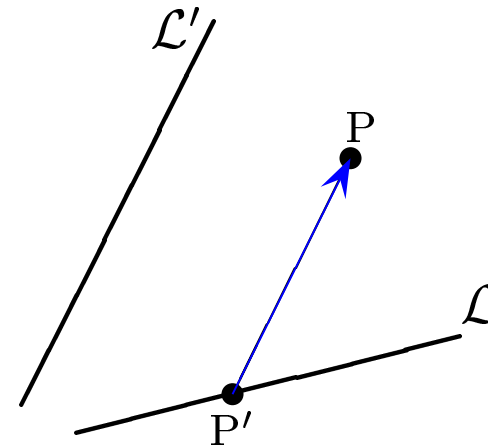
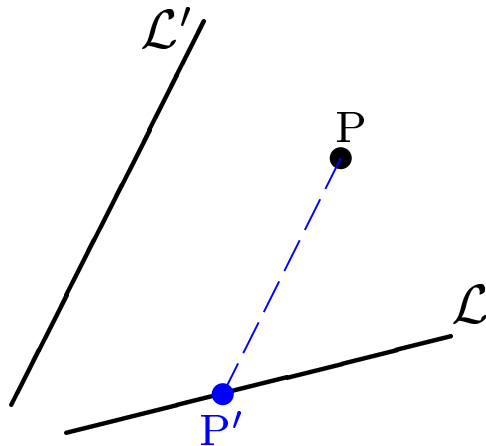


## II.3 Principal Axes of a Cloud

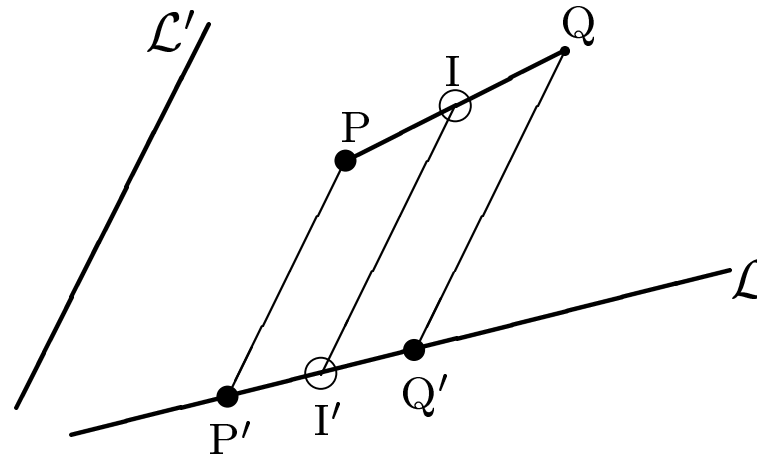
### Projection of a cloud

$P'$  = projection of point  $P$  onto  $\mathcal{L}$  along  $\mathcal{L}'$

$\overrightarrow{P'P}$  = residual deviation

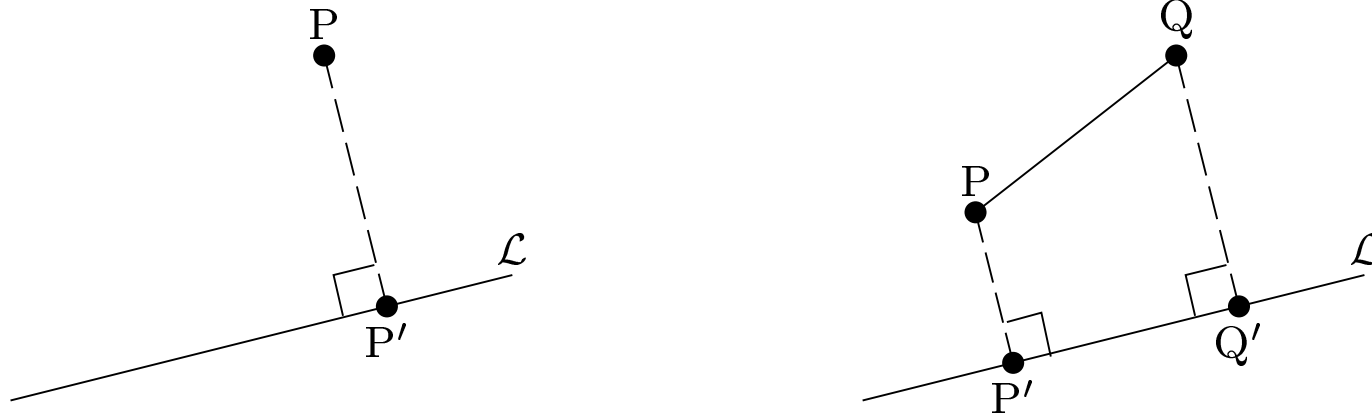


If  $I$  is the midpoint of  $P$  and  $Q$ , the projection  $I'$  of  $I$  on  $\mathcal{L}$  is the midpoint of  $P'$  and  $Q'$ .



*Mean point property:* The mean point is preserved by projection.

*Orthogonal projection:*  $PP'$  is perpendicular to  $\mathcal{L}$ .



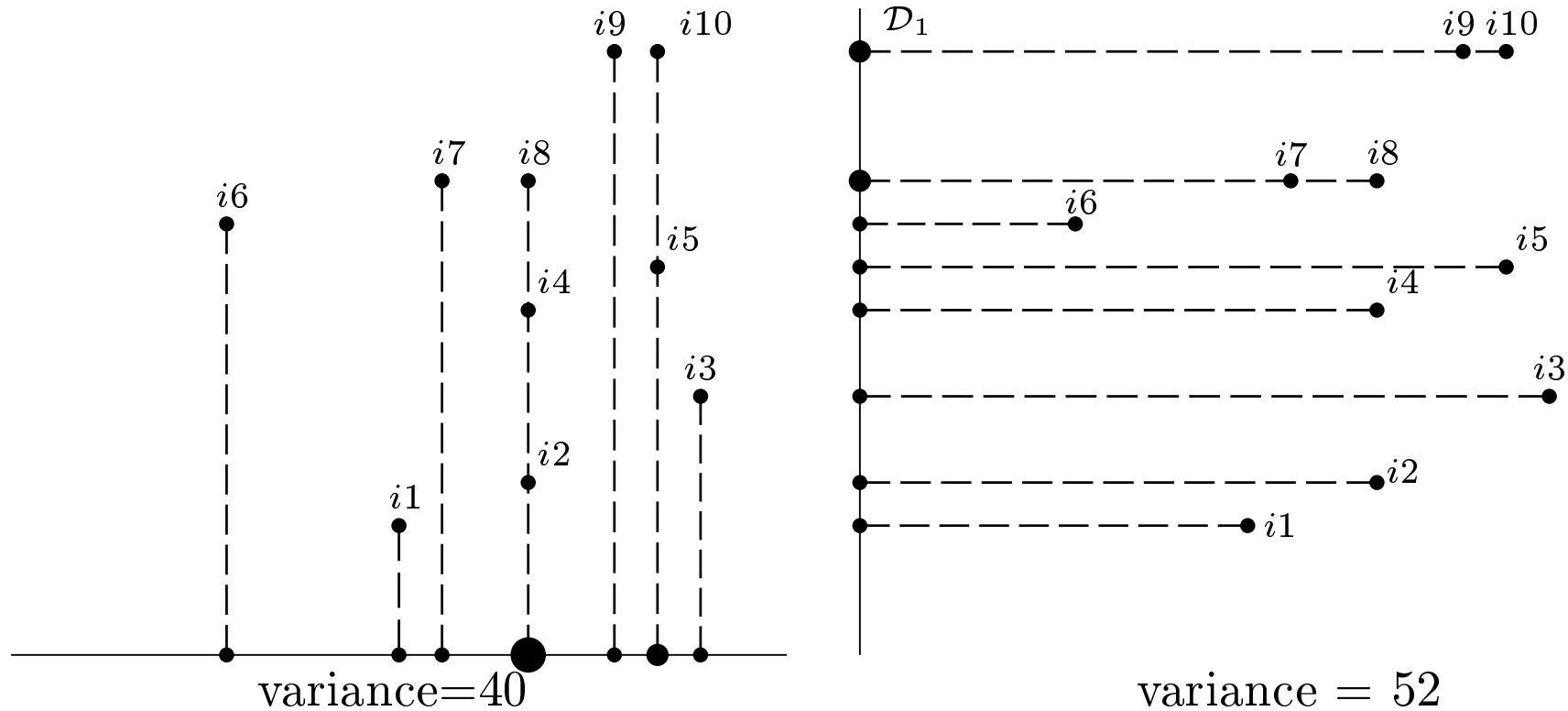
*Property:* the orthogonal projection contracts distances

$$P'Q' \leq PQ$$

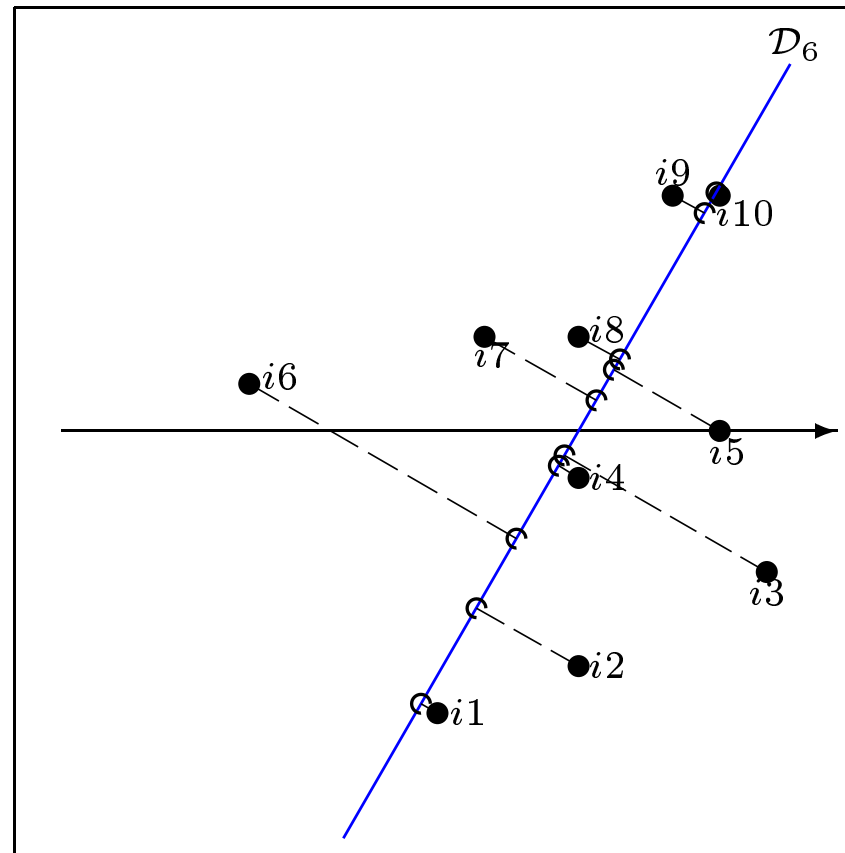
therefore

variance of a projected cloud  $\leq$  variance of the initial cloud.

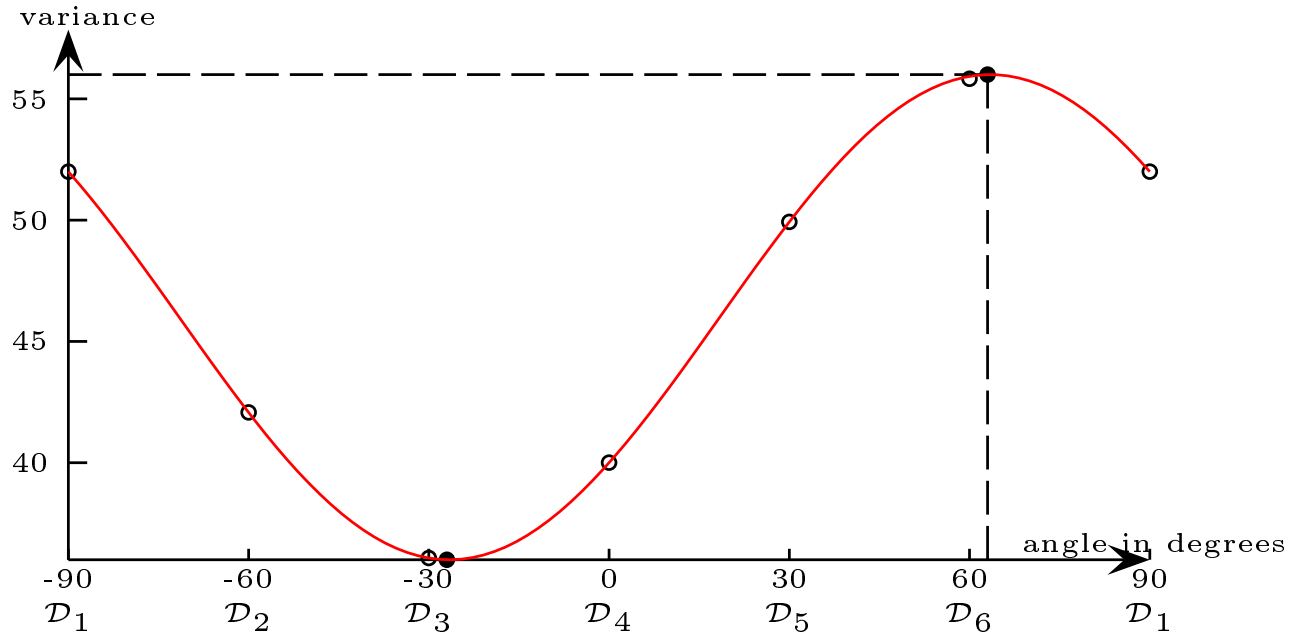
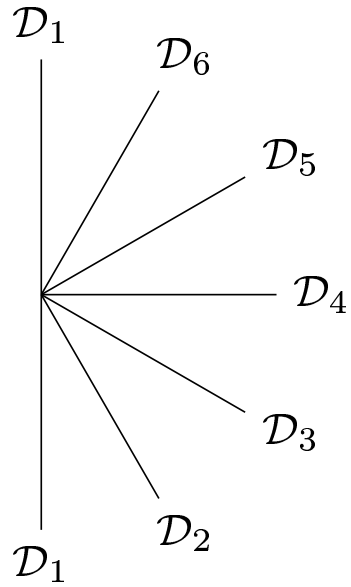
## Projected clouds on several lines



*Orthogonal additive decomposition* : the sum of the variances of projected clouds onto perpendicular lines is the variance of initial cloud:  $40 + 52 = 92$ .



Projection onto an oblique line (60 degrees) : variance = 55.9



	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_1$
Variance	52	42.1	36.1	40.0	49.9	55.9	52

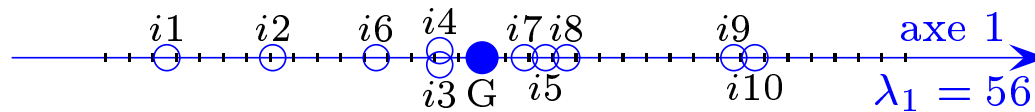
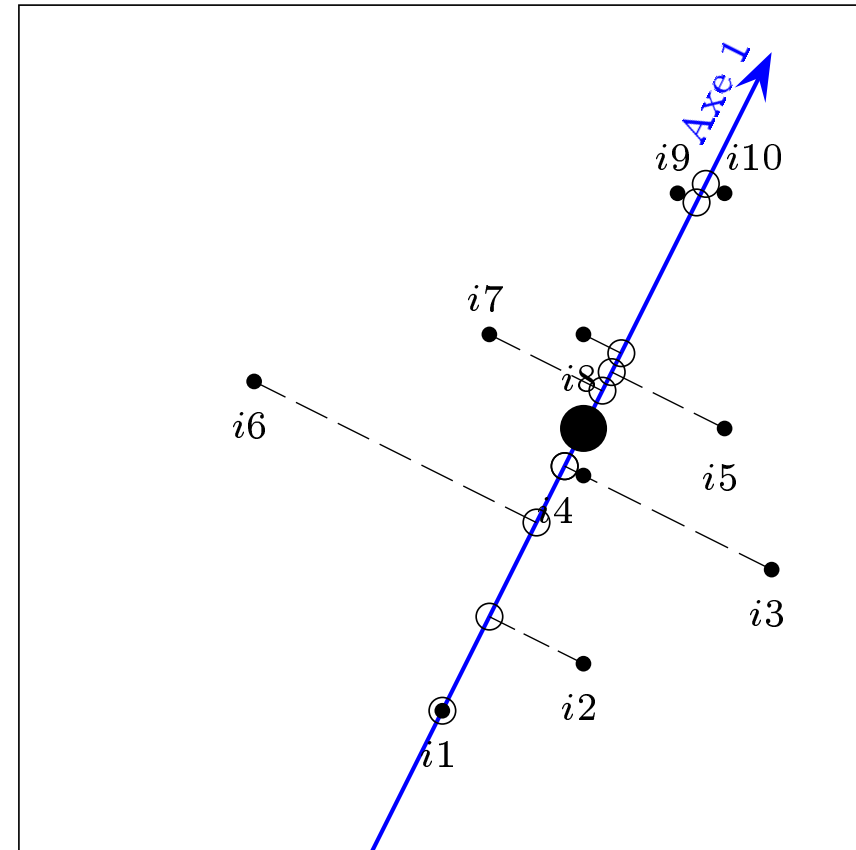
The directed line such as the variance of the projected cloud is maximum is called *first principal axis*.

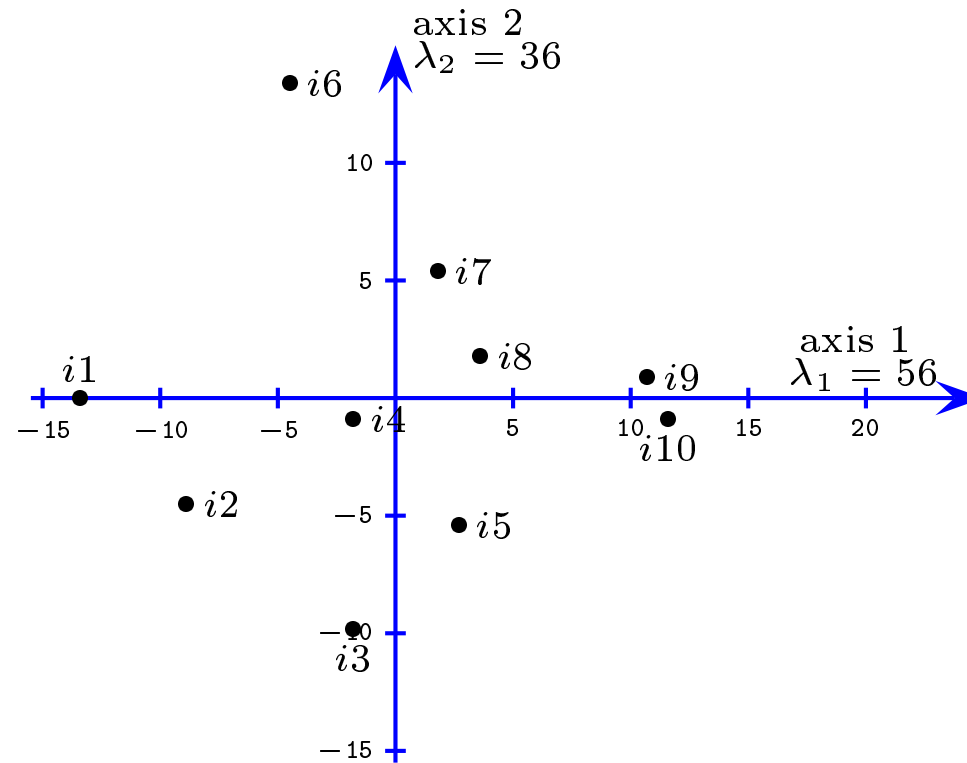
Projected cloud = *first principal cloud*,

its variance ( $\lambda$ ) = *variance of axis*.

The first principal cloud is the best fitting of the initial cloud by an unidimensional cloud in the sense of orthogonal least squares

Here,  $\alpha = 63^\circ$ ,  $\lambda_1 = 56$ .





Principal representation of the cloud (plane 1-2).

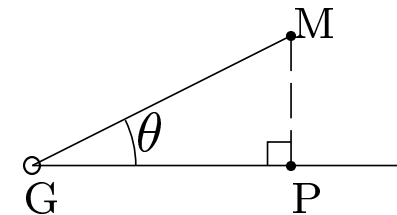


- *Quality of fit of an axis* or variance rate  $\frac{\lambda}{V_{\text{nuage}}}$
- The principal coordinates of points define *principal variables* with mean = 0 and variance =  $\lambda$  (eigenvalue)  
The principal variables are uncorrelated

- *Contribution of point  $i$  to an axis* :  $\text{Ctr} = \frac{p(y)^2}{\lambda}$
- *Quality of representation of a point onto an axis* :

$$\cos^2 \theta = \frac{GP^2}{GM^2}$$

*Example*: for  $i_2$ ,  $\cos^2 \theta = \frac{(-8.94)^2}{100} = 0.80$



- *Reconstitution of distances* :

$$d^2(i_1, i_2) = (-13.4 + 8.9)^2 + (0 - 4.47)^2 = 4.23 = (6.3)^2$$

## Results of the analysis

$\lambda_1 = 56$  (variance of axis 1, eigenvalue).

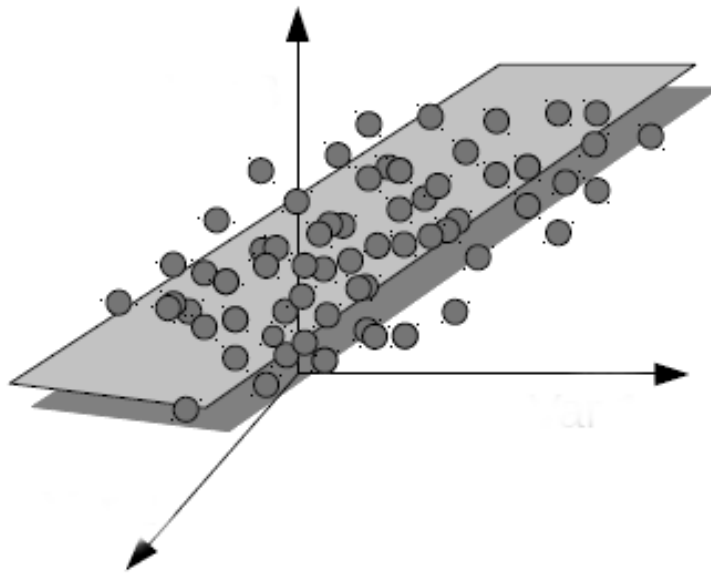
$$\text{Variance rate} : \frac{\lambda_1}{V_{\text{nuage}}} = \frac{56}{92} = 61\%$$

		<i>Results for axis 1</i>			<i>Results for axis 2</i>		
		$\lambda_1 = 56$			$\lambda_1 = 36$		
	$p_i$	Coordinates	Ctr (%)	squared cosines	Coordinates	Ctr (%)	squared cosines
<i>i1</i>	0.1	-13.41	32.1	1.00	0.00	0	0.00
<i>i2</i>	0.1	-8.94	14.3	0.80	+4.47	5.6	0.20
<i>i3</i>	0.1	-1.79	0.6	0.03	+9.84	26.9	0.97
<i>i4</i>	0.1	-1.79	1.3	0.80	+0.89	0.2	0.20
<i>i5</i>	0.1	+2.68	3.6	0.20	+5.37	8	0.80
<i>i6</i>	0.1	-4.47	3.6	0.10	-13.42	50.0	0.90
<i>i7</i>	0.1	+1.79	0.6	0.10	-5.37	8	0.90
<i>i8</i>	0.1	+3.58	2.3	0.80	-1.79	0.9	0.20
<i>i9</i>	0.1	+10.73	20.6	0.99	-0.89	0.2	0.01
<i>i10</i>	0.1	+11.63	24.1	0.99	+0.89	0.2	0.01

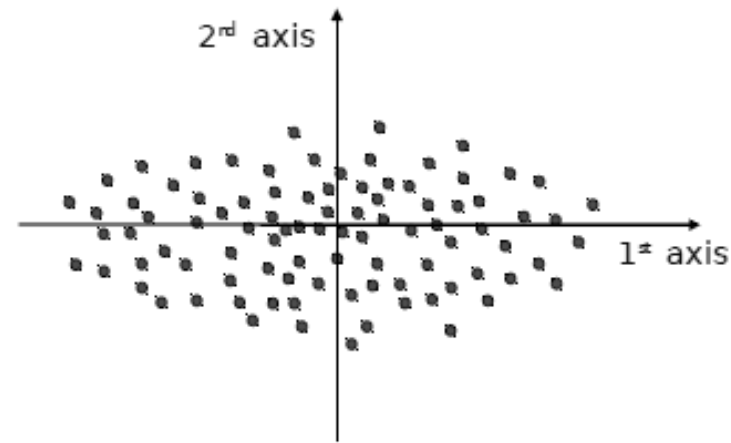
## II.4 From a Plane Cloud to a Higher Dimensional Cloud

**Heredity property:** the plane that best fits the cloud is the one determined by the first two axes.

High dimensional cloud.



Low dimensional projection.



## Properties

- $\sum_{\ell=1}^L \lambda_{\ell} = V_{\text{nuage}}$  ( $L$  denotes the dimensionality of the cloud).
- The principal axes are pairwise orthogonal.
- Each axis can be directed arbitrarily.
- The principal variables corresponding to distinct eigenvalues are uncorrelated.